On the quadratic integral equations and their applications

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A R T I C L E I N F O

Article history:
Received 22 May 2010
Received in revised form 27 July 2011
Accepted 29 July 2011

Keywords:
Fractional calculus
Existence theorem
Quadratic integral equation
Chandrasekhar’s integral equation

A B S T R A C T

The class of quadratic integral equations contains, as a special case, numerous integral equations encountered in the theory of radiative transfer, the queuing theory, the kinetic theory of gases and the theory of neutron transport. As a pursuit of this, in the following pages, sufficient conditions are given for the existence of positive continuous solutions to some possibly singular quadratic integral equations. Meanwhile, we prove the existence of maximal and minimal solutions of our problems. The method used here depends on both Schauder and Schauder–Tychonoff fixed point principles. Unlike all previous contributions of the same type, no assumptions in terms of the measure of noncompactness were imposed on the nonlinearity of the given functions. As far as we know, the approach presented in this paper, in particular, the discussion of the existence of maximal and minimal solutions to the quadratic integral equations was never applied in the field of the quadratic integral equations and so is new.

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1. Introduction and preliminaries

Integral equations of various types play an important role in many branches of functional analysis and in their applications in physics, economics and other fields. In particular, quadratic integral equations have many useful applications in describing numerous events and problems of the real world. For example, quadratic integral equations are often applicable in the theory of radiative transfer, the kinetic theory of gases, the theory of neutron transport, the queuing theory and the traffic theory (see e.g. [1–4]).

The aim of this paper is two-fold. On the one hand, we establish a sufficient condition to ensure the existence of positive continuous solutions to the possibly singular quadratic integral equation

\[ x(t) = H(t, x(t)) + x(t) \int_0^1 k(t, s)\psi(s)(f(x(s)) + g(x(s))) \, ds, \quad t \in [0, 1]. \quad (1) \]

By singularity, we mean that the possibility of \( g(0) \) being undefined is permitted. For completeness, we consider, as a special case of problem (1), the following quadratic integral equation of fractional type

\[ x(t) = H(t, x(t)) + x(t)^\alpha \psi(s)(f(x(t)) + g(x(t))), \quad t \in [0, 1], \quad \alpha \in (0, 1). \quad (2) \]

On the other hand, we deal with the possibly singular quadratic integral equation

\[ x(t) = H(t, x(t)) + x(t) \int_0^1 k(t, s)\psi(s)(f(x(s)) + g(x(s))) \, ds, \quad t \in [0, 1]. \quad (3) \]

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The existence of continuous solutions to the Chandrasekhar's integral equation [5]

\[ x(t) = 1 + x(t) \int_{0}^{1} \frac{t}{t + s} \varphi(s) x(s) \, ds, \]  

will be given as an application.

To encompass the full scope of our paper, we investigate the problem of the existence of solutions of the quadratic integral equation

\[ x(t) = H(t, x(t)) + x(t) \int_{0}^{\infty} k(t, s) \varphi(s) (f(x(s)) + g(x(s))) \, ds, \quad t \in [0, \infty). \]  

In our investigations, we assume that \( f : [0, \infty) \to [0, \infty) \) and \( g : (0, \infty) \to [0, \infty) \) are nonlinear continuous functions such that \( f \) is nondecreasing while \( g \) is nonincreasing and possibly singular, that is, the possibility of \( g(0) \) being undefined is allowed. By placing appropriate conditions on \( H, k \) and \( \varphi \), we use Schauder’s and Schauder–Tychonoff fixed point theorems to prove the existence of a continuous solution \( x \) to the above problems such that \( 0 < \mu \leq x(t) \leq \gamma \) for some \( 0 < \mu < \gamma \).

It is worth mentioning that, the theory of quadratic integral equations with nonsingular kernels has received a lot of attention. Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels (see e.g. [16–10] and the references therein). However, in most of the above literature, the main results are realized with the help of the technique associated with the measures of noncompactness and a fixed point theorem of Darbo type. This approach seems to be too restrictive. Further, in most of the above investigations, the function \( H \) is assumed to be independent of the unknown function and the function \( x \to f(x) + g(x) \) is assumed to be nondecreasing. Some additional assumptions in terms of the measure of noncompactness were imposed on \( H \). In comparison with the existence results in these references, our assumptions are more natural. The most interesting point is that the function \( g \) may be singular at \( x = 0 \). Also, we drop the requirement that \( x \to f(x) + g(x) \) is monotonic. Further, instead of using the technique associated with Darbo’s fixed point theorem, we proceed in a different way by making up the technique associated with Schauder’s fixed point theorem. Let us notice that the approach presented in this paper was never applied in the field of quadratic integral equations.

Let \( I = [0, 1] \). According to the convention \( L^p(I), \ 1 \leq p \leq \infty \) will denote the Banach space of real-valued measurable functions \( x \) defined over \( I \) with \( |x|^p \) as a Lebesgue integrable function on the interval \( I \), and \( L^\infty(I) \) the Banach space of real-valued essentially bounded and measurable functions defined over \( I \). We will let \( \| \cdot \|_p \) denote the usual norm in the Banach space \( L^p(I) \). Let \( C(I) \) be the space of continuous functions on the interval \([0, 1]\) with the usual sup-norm. Denote by \( BC[0, \infty) \) the normed space of bounded, continuous functions defined on \([0, \infty)\). In addition, we denote by \( C[0, \infty) \) the Fréchet space of all continuous functions on \([0, \infty)\). For \( y \in C[0, \infty) \), we define, for each \( m \in \mathbb{N} := \{1, 2, \ldots \} \), the seminorm \( \rho_m(y) \) by \( \rho_m(y) := \sup_{t \in [0, m]} |y(t)|. \) Recall that a subset \( C \) of \( C[0, \infty) \) is bounded if a positive function \( a \in C[0, \infty) \) exists such that \( |y(t)| \leq a(t) \), for all \( t \in [0, \infty) \) and \( y \in C \). Furthermore, recall that a family \( A \) of \( C[0, \infty) \) is relatively compact if and only if for each \( T > 0 \), the restrictions to \( [0, T] \) of all functions from \( A \) form an equicontinuous and uniformly bounded set.

The following fixed point theorems play a major role in our analysis.

**Theorem 1.1 (Schauder Fixed Point Theorem).** Let \( Q \) be a nonempty, convex, compact subset of a Banach space \( X \), and \( T : Q \to Q \) be a continuous map. Then \( T \) has at least one fixed point in \( Q \).

**Theorem 1.2 (Schauder–Tychonoff Fixed Point Theorem).** Let \( Q \) be a nonempty, convex, closed subset of a locally convex, Hausdorff space \( X \). Assume that \( T : Q \to Q \) is continuous and \( T(Q) \) is relatively compact in \( X \). Then \( T \) has at least one fixed point in \( Q \).

## 2. Existence of positive continuous solutions

In this section, we prove the existence of positive continuous solutions for Eq. (1). To facilitate our discussion, let us first state the following assumptions

1. \( H : [0, 1] \times [0, \infty) \to [0, \infty) \) is continuous and satisfies the following conditions
   - (a) \( 0 < \mu < \gamma \) exists such that \( H(t, x) \geq \mu \) holds for every \( x \geq \mu \) and \( \gamma \geq 2 \max_{t \in [0, 1]} H(t, \mu) \).
   - (b) There exists a function \( b \in C[0, 1] \) and nondecreasing functions \( c_i : [\mu, \infty) \to \mathbb{R}^+ \), \( i = 1, 2 \), such that
     \[
     |H(t, x) - H(s, y)| \leq c_1(r)|b(t) - b(s)| + c_2(r)|x - y|,
     \]
     for all \( x, y \in [\mu, r] \) and \( t, s \in [0, 1] \).
2. \( f : [0, \infty) \to [0, \infty) \) is continuous and nondecreasing.
3. \( g : (0, \infty) \to [0, \infty) \) is continuous and nonincreasing.
4. \( 0 \leq \varphi \in L^p(I) \).
Let \( k(t, s) = k(t, s) \in L^2[0, 1] \) for each \( t \in [0, 1] \) and the map \( t \to k(t, s) \) is continuous from \([0, 1]\) to \( L^2[0, 1] \).

(6) 
\[
2 \left( c_2(\gamma') + (f(\gamma) + g(\mu)) \sup_{t \in [0, 1]} \int_0^t k(t, s) \varphi(s) \, ds \right) \leq 1.
\]

Remark 2.1. We are able to relax Assumption (6), if we assume that the function \( H \) is independent of \( x \); in this case, we deduce that \( c_2(\cdot) \equiv 0 \) and then Assumption (6) takes the form
\[
2 \left( f(\gamma) + g(\mu) \right) \sup_{t \in [0, 1]} \int_0^t k(t, s) \varphi(s) \, ds \leq 1.
\]

In addition, if \( f \equiv 0 \), \( g(x) = \frac{1}{2} \) and \( H(t, x) = e^{5t} \), Assumption (6) takes the form
\[
\left( \sup_{t \in [0, 1]} \int_0^t k(t, s) \varphi(s) \, ds \right) \leq \frac{\mu}{2}, \quad \mu \leq e^5.
\]

Remark 2.2. We also remark that, if we replace the function \( x \to f(x) + g(x) \) by \( x \to f(x)g(x) \), then it can be easily shown that the main results of our paper remain valid provided we replace Assumption (6) with the following one
\[
2 \left( c_2(\gamma') + f(\gamma)g(\mu) \sup_{t \in [0, 1]} \int_0^t k(t, s) \varphi(s) \, ds \right) \leq 1.
\]

Furthermore, if \( H \) is independent of \( x \) and if \( g(x) = [\psi(x)]^{-1} \), where \( \psi : [0, \infty) \to [0, \infty) \) is continuous and nondecreasing, then Assumption (6) takes the form
\[
\left( \sup_{t \in [0, 1]} \int_0^t k(t, s) \varphi(s) \, ds \right) \leq \frac{\psi(\mu)}{2f(\gamma)}, \quad f(\gamma) \neq 0.
\]

In account of the above remarks, we can see that Assumption (6) is not too restrictive.

Again, let us pay attention to Assumptions (1)–(6). In the view of this assumption, we see that the function \( f + g \) occurring in the integral equations (1), (3) and (5) are not necessarily monotonic with respect to \( x \). Moreover, the function \( H \) depends on the unknown function and no monotonicity conditions imposed on the nonlinearity of \( H \). This means that the previous results of this type (see e.g. [16–10]) are not applicable to these equations. Finally, let us mention that in our case, the interval \( I = [0, 1] \) may be replaced by an arbitrary interval \( I = [a, b] \).

As an example of natural mappings satisfying Assumptions (1)–(6), we have the following.

Example 2.1. Let
\[
H(t, x) = \frac{1 + t}{10} + e^{-t} \frac{x^2}{30}.
\]

Set \( \mu = 0.1 \); then for every \( x \geq \mu \) and every \( t \in [0, 1] \), we have \( H(t, x) > 0.1 \) and \( 2H(t, 0.1) < 0.5 \).

Now, let \( \gamma = 0.5 \); then \( \gamma > \mu \) and for any \( x, y \in [\mu, r], \ r > 0 \) and every \( t, s \in [0, 1] \), we have
\[
|H(t, x) - H(s, y)| \leq \frac{1}{10} |t - s| + \frac{1}{30} \left| e^{-t} x^2 - e^{-s} y^2 \right| + \frac{1}{30} \left| e^{-t} y^2 - e^{-s} y^2 \right|.
\]

By the standard mean value theorem, we deduce
\[
|H(t, x) - H(s, y)| \leq \frac{1}{10} |t - s| + \frac{2|x + y|}{30} |x - y| + \frac{r^2}{30} |e^{-t} - e^{-s}|
\]
\[
\leq \frac{1}{10} |t - s| + \frac{r}{30} |x - y| + \frac{r^2 e^{-s}}{30} |t - s|, \quad \gamma \in (0, 1).
\]

Then it is clear that this choice of \( H \) satisfies Assumptions (1) with \( \mu = 0.1, \ \gamma = 0.5, b(t) = t, \ t \in [0, 1] \) and
\[
c_1(r) = \frac{1}{10} + \frac{r^2}{30}, \quad c_2(r) = \frac{2r}{30}.
\]

Also, we note that \( c_2(\gamma) = c_2(0.5) = 0.03 < 0.5 \).

Now, we are prepared to formulate and prove the following existence result.

Theorem 2.1. Assume that Assumptions (1)–(6) be satisfied. Then Eq. (1) has at least one solution \( x \in C[0, 1] \) such that \( 0 < \mu \leq x(t) \leq \gamma, \ t \in [0, 1] \).
To solve Eq. (1), it is necessary to find a fixed point of the operator $T : C[0, 1] \to C[0, 1]$ defined by

$T(x)(t) := H(t, x(t)) + x(t) \int_0^t k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds, \quad t \in [0, 1].$ \hspace{1cm} (9)

Let $Q$ be the subset of the space $C[0, 1]$ defined as follows:

$Q = \left\{ x \in C[0, 1] : \mu \leq x(t) \leq \gamma, \forall t \in [0, 1] \text{ and } \forall t_1, t_2 \in [0, 1], \right\}$

we have $|x(t_1) - x(t_2)| \leq \frac{1}{1 - K^*} \left[ b_{1,2} + \|f(\gamma) + g(\mu)\| \right] k_{1,2}.$

where

$K^* := \left( c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{t \in [0, 1]} \int_0^t k(t, s)\varphi(s) \, ds \right),$

$k_{1,2} := \left\| \int_{t_1}^{t_2} k(t, s)\varphi(s) \, ds \right\|, \quad b_{1,2} := c_1(\gamma) |b(t_2) - b(t_1)|.$

Note that $Q$ is a nonempty (since $\mu < \gamma$), closed, bounded, convex and equicontinuous (hence compact) subset of $C[0, 1].$

We claim $T : Q \to Q.$ To prove our claim, first we note that, $\varphi(\cdot) \in L^p([a, b]) \text{ and } k(t, \cdot) \in C([a, b]). \quad t \in I.$ Therefore, the function $f(x(\cdot))$ and $g(x(\cdot))$ are compositions of this mapping with $f$ and $g,$ respectively, and thus, for each $x \in Q,$ $f(x(s)) : [0, 1] \to f(\mu), f(\gamma)$ and $g(x(s)) : [0, 1] \to g(\gamma), g(\mu)$ are continuous; hence the operator $T$ makes sense. Now, let $x \in Q$ and $t_1, t_2 \in [0, 1]$ with $t_1 \leq t_2.$ In the view of our assumptions we obtain

$|T(x(t_2)) - T(x(t_1))| \leq |H(t_2, x(t_2)) - H(t_1, x(t_1))|$

$+ \left| x(t_2) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds - x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds \right|$

$\leq c_1(\|x\|) |b(t_2) - b(t_1)| + c_2(\|x\|) |x(t_2) - x(t_1)|$

$+ \left| x(t_2) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds - x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds \right|$

$\leq c_1(\|x\|) |b(t_2) - b(t_1)| + c_2(\|x\|) |x(t_2) - x(t_1)|$

$+ \left| x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds - x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds \right|$

$\leq c_1(\|x\|) |b(t_2) - b(t_1)| + c_2(\|x\|) |x(t_2) - x(t_1)|$

$+ \left| x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds - x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds \right|$

$\leq c_1(\|x\|) |b(t_2) - b(t_1)| + c_2(\|x\|) |x(t_2) - x(t_1)|$

$+ \left| x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds - x(t_1) \int_0^{t_1} k(t, s)\varphi(s)(f(x(s)) + g(x(s))) \, ds \right|$

$\leq c_1(\|x\|) |b(t_2) - b(t_1)| + c_2(\|x\|) |x(t_2) - x(t_1)|$

Thus we have

$|T(x(t_2)) - T(x(t_1))| \leq b_{1,2} + K^* \left[ \|x(t_2) - x(t_1)\| + \frac{(f(\gamma) + g(\mu)) \|x\|}{K^*} \right].$ \hspace{1cm} (10)

The above inequality and our assumptions yield

$|T(x(t_2)) - T(x(t_1))| \to 0 \quad \text{as} \quad t_2 \to t_1;$

then $T$ is uniformly continuous in $[0, 1].$ We claim that $T : Q \to Q$ is continuous. Once our claim is established, according to Schauder's fixed point theorem, $T$ has a fixed point in $Q.$ It remains to prove our claim by showing that $T$ maps $Q$ into
we have
\[ |Tx(t_2) - Tx(t_1)| \leq \frac{1}{1 - K^*} \left[ b_{1,2} + (f(\gamma) + g(\mu))\gamma \left( \| k_{t_2} - k_{t_1} \|_q \| \varphi \|_p + \int_{t_1}^{t_2} k_{t_2}(s)\varphi(s) \, ds \right) \right]. \]

Moreover, we have
\[
Tx(t) \geq \min_{\tau \in [0,1]} H(t, x(t)) + \mu(f(\mu) + g(\mu)) \int_0^t k(t, s)\varphi(s) \, ds \geq \mu.
\]

Further, for every \( t \in [0, 1] \) and \( x \in Q \), we have
\[
\|Tx\| \leq \|H(\cdot, x(\cdot))\| + (f(\gamma) + g(\mu)) \|x\| \left( \sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \right)
\leq \|H(\cdot, x(\cdot)) - H(\cdot, \mu)\| + \|H(\cdot, \mu)\| + (f(\gamma) + g(\mu)) \|x\| \left( \sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \right)
\leq c_2(\|x\|) \|x\| + \max_{t \in [0,1]} H(t, \mu) + (f(\gamma) + g(\mu)) \|x\| \left( \sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \right).
\]

Thus
\[
\|Tx\| \leq \max_{t \in [0,1]} H(t, \mu) + \gamma \left( c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \right) \leq \frac{\gamma}{2} + \frac{\gamma}{2} = \gamma. \tag{11}
\]

Consequently, \( \mu \leq Tx(t) \leq \gamma, \ t \in [0, 1] \). Hence \( T : Q \to Q \) is well-defined. Next, let \( x_n \to x \) in \( Q \). For any \( t \in [0, 1] \), we have
\[
|Tx_n(t) - Tx(t)| \leq |H(t, x_n(t)) - H(t, x(t))|
\leq |x_n(t)\int_0^t k(t, s)\varphi(s)(f(x_n(s)) + g(x_n(s))) \, ds - x(t)\int_0^t k(t, s)\varphi(s)(f(x_n(s)) + g(x_n(s))) \, ds|
\leq c_2(\|x_n - x\|)(f(\gamma) + g(\mu)) \left( \sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \right)
\leq \gamma \left( \sup_{\tau \in [0,1]} |f(x_n(t)) - f(x(t))| + \sup_{\tau \in [0,1]} |g(x_n(t)) - g(x(t))| \right) \sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds.
\]

We have, therefore, shown that \( T : Q \to Q \) is a continuous operator; hence by Schauder’s fixed point theorem, \( T : Q \to Q \) has a fixed point. Consequently, the integral equation (1) has a positive continuous solution. \( \square \)

Now, we introduce an example to illustrate the result contained in Theorem 2.1.

**Example 2.2.** Consider the following functional integral equation
\[
x(t) = \frac{1}{\ln(5)} \ln \left( \frac{20 + \sqrt{x(t)}}{1 + t} \right) + \frac{x(t)}{5} \int_0^t \frac{s}{1 + t + s^2} \left[ \ln(1 + x(s)) + \frac{1}{4\sqrt{x(s)}} \right] \, ds, \quad t \in [0, 1]. \tag{12}
\]

Observe that the above equation is a special case of Eq. (1) if we put
\[
\varphi(s) = 0.2s, \quad f(x) = \ln(1 + x), \quad g(x) = \frac{1}{4\sqrt{x}} \quad \text{and}
\]
\[
k(t, s) = \frac{1}{1 + t + s^2}, \quad H(t, x) = \frac{1}{\ln(5)} \ln \left( \frac{20 + \sqrt{x}}{1 + t} \right).
\]

In what follows, we show that functions involved in Eq. (12) satisfy Assumptions (1)–(6) with \( p = 1, \ q = \infty \). In fact, we have
\[
\sup_{\tau \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \leq \sup_{\tau \in [0,1]} \int_0^t 0.2s \frac{1}{1 + s^2} \leq 0.1 \ln(4) < 0.11.
\]

If we set \( \mu = 0.1 \), then for every \( x \geq \mu \) and every \( t \in [0, 1] \), we have
\[
H(t, x) \geq \frac{\ln(10.005)}{\ln(5)} \simeq 1.6, \quad 2H(t, \mu) \leq 2\frac{\ln(20.01)}{\ln(5)} = 3.4289041 < 4.
\]
Now, let \( \gamma = 4 \), and assume that \( x, y \in [\mu, r] \), \( r > 0 \) and \( t, s \in [0, 1] \), we have

\[
|H(t, x) - H(s, y)| \leq \frac{1}{\ln(5)} \left[ |\ln(1 + t) - \ln(1 + s)| + |\ln(20 + \sqrt{x}) - \ln(20 + \sqrt{y})| \right].
\]

Using the standard methods of differential calculus we can easily prove that

\[
|\ln(20 + \sqrt{x}) - \ln(20 + \sqrt{y})| \leq \frac{1}{2\sqrt{5}} \frac{|\sqrt{x} - \sqrt{y}|}{20 + \sqrt{x}} \frac{1}{\sqrt{x} + \sqrt{y}} |x - y|.
\]

Since \( \xi > 20.01 \), then \( \frac{1}{2\sqrt{\xi}(20 + \sqrt{\xi})} < 0.0046 \). Consequently,

\[
|H(t, x) - H(s, y)| \leq \frac{1}{\ln(5)} |\ln(1 + t) - \ln(1 + s)| + 0.0023 |x - y|.
\]

Then it is clear that this choice of \( H, f, k, \varphi \) and \( g \) satisfy Assumptions (1)–(5) with \( \mu = 0.1 \), \( \gamma = 4 \), \( b(t) = \ln(1 + t) \) and \( c_1(r) = \frac{1}{\xi} \), \( c_2(r) = 0.0046 \). Finally, it remains to verify that Assumption (6) is satisfied. Obviously,

\[
2 \left( c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{t \in [0, 1]} \int_0^t k(t, s)\varphi(s) \, ds \right) \leq 0.0092 + 0.22(\ln(4) + 0.633) \leq 0.8 < 1.
\]

Simultaneously, on the basis of Theorem 2.1 we conclude that the integral equation (12) has at least one continuous solution \( x \) with \( 0.1 \leq x(t) \leq 4 \).

**Example 2.3.** In the view of the above example and Remark 2.2, it can be easily seen that the quadratic integral equation

\[
x(t) = \frac{1}{\ln(5)} \ln \left( \frac{20 + \sqrt{x(t)}}{1 + t} \right) + \frac{x(t)}{5} \int_0^t \frac{s}{1 + t + s^2} \left[ \frac{\ln(1 + x(s))}{4x(s)} \right] \, ds, \quad t \in [0, 1],
\]

has at least one continuous solution \( x \) with \( 0.1 \leq x(t) \leq 4 \).

Also, in the view of Remark 2.2, we have the following example.

**Example 2.4.** Let

\[
f(x) = \ln \left( 1 + \sqrt[5]{0.5x} \right), \quad g(x) = \frac{1}{2x} \quad \text{and} \quad H(t, x) = e^{5t}.
\]

and consider the problem

\[
x(t) = e^{5t} + x(t) \int_0^t \frac{t \exp \left( \frac{-1}{t + s} \right)}{(t + s)^2} \varphi(s) \left[ \ln \left( 1 + \sqrt[5]{0.5x(s)} \right) \right] \, ds, \quad t \in [0, 1].
\]

It is easy to check, in this situation, that Assumptions (1)–(5) are satisfied for \( \mu = e^5 \), \( \gamma = 2e^6 \), \( p = 1 \) and \( c_2(t) \equiv 0 \). Moreover, it may be verified that \( 2(f(\gamma)g(\mu)) = \frac{\ln(2e)}{e^5} \) and

\[
2(f(\gamma)g(\mu)) \sup_{t \in [0, 1]} \int_0^t k(t, s)\varphi(s) \, ds \leq \frac{\ln(2e)}{e^5} \sup_{t \in [0, 1]} \int_0^t \frac{t}{(t + s)^2} \exp \left( \frac{-1}{t + s} \right) \varphi(s) \, ds \leq \frac{\ln(2e)}{e^{5.5}} \|\varphi\|_{\infty}.
\]

Thus, to ensure that Assumption (6) will be valid, it is enough to take \( \varphi \in L^\infty \) such that \( \|\varphi\|_{\infty} \leq \frac{e^{5.5}}{\ln(2e)} \).

**Example 2.5.** Consider the following functional integral equation

\[
x(t) = \frac{1 + t}{10} + e^{-t} \frac{x^2(t)}{30} + \frac{x(t)}{5} \int_0^t \frac{s}{1 + t + s^2} \left[ \cosh(x(s)) \right] \, ds, \quad t \in [0, 1].
\]

The above equation is also a special case of Eq. (1), if we put

\[
\varphi(s) = 0.2s, \quad f(x) = \frac{e^x}{2}, \quad g(x) = \frac{e^{-x}}{2} \quad \text{and},
\]

\[
k(t, s) = \frac{1}{1 + t + s^2}, \quad H(t, x) = \frac{1 + t}{10} + e^{-t} \frac{x^2}{30} + \frac{x(t)}{5}.
\]
Observe Examples 2.1 and 2.2. Invoking the obtained formulas expressing $c_2$, and the estimates of the constants $\mu$, $\gamma$, we see that
\[
2 \left( c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{t \in [0,1]} \int_0^t k(t, s)\varphi(s) \, ds \right) \leq 0.04 + (1.13)0.22 \leq 0.29 < 1.
\]
Thus, on the basis of Theorem 2.1 we conclude that the integral equation (13) has at least one continuous solution $x$ with $0.1 \leq x(t) \leq 0.3$.

3. Maximal and minimal solutions

**Definition 3.1.** Let $m$ be a solution of the integral equation (1) in $[0, 1]$; then $m$ is said to be a maximal solution of (1) if, for every solution $x$ of (1) existing on $[0, 1]$, the inequality $x(t) \leq m(t), \ t \in [0, 1]$, holds. A minimal solution may be defined similarly by reversing the last inequality.

**Theorem 3.1.** Suppose that Assumptions (1)–(6) are satisfied with $f(\cdot) = 0$ and with a constant function $c_2$. If there exists $\vartheta(\cdot) \in L^q$ such that for any $t \in [0, 1], k(t, s) \leq \vartheta(s), s \in [0, 1]$. Then there exists either maximal or minimal solutions of the integral equation (1) in the space $C([0, 1], (0, \infty))$.

**Remark 3.1.** If $H$ is independent of $x$, then the requirement of Theorem 3.1 that $c_2$ is a constant function is automatically satisfied where $c_2 = 0$.

**Proof.** Consider the integral equation
\[
x(t) = \frac{1}{n} + H(t, x(t)) + x(t) \int_0^t k(t, s)\varphi(s)x(s) \, ds, \quad t \in [0, 1], \ n \in \mathbb{N}.
\] (14)

In the view of our assumptions with $H_n(t, x) := \frac{1}{n} + H(t, x)$, it is clear that
\[
H_n(t, x) \geq \mu, \quad t \in [0, 1] \quad \text{and} \quad 2H_n(t, \mu) \leq \frac{2}{n} + \gamma \leq 2 + \gamma := \gamma^*.
\]

Since $c_2$ is a constant function, then the assumptions of Theorem 2.1 hold; therefore, the integral equation (14) has at least one solution $x_n \in C[0, 1]$, such that the inequality $\mu \leq x_n(t) \leq \gamma^*, \ t \in [0, 1]$ holds for every $n \in \mathbb{N}$.

Now define the function $u : [0, 1] \rightarrow \mathbb{R}^+$ by
\[
u(t) := \int_0^t k(t, s)\varphi(s)x(s) \, ds.
\]

As in the proof of Theorem 2.1, $u \in C([0, 1], \mathbb{R}^+)$. We will show that $u(0) = 0$. To see this, we observe that
\[
u(t) \leq g(\mu) \int_0^t \varphi(s) \, ds \rightarrow 0 \quad \text{as} \ t \rightarrow 0.
\]

Since $u$ is continuous, one can deduce that $u(0) = 0$.

We claim that the family of functions $\{x_n\} \in \text{relatively compact on} [0, 1].$ To see this, note that $\{x_n\}$ is uniformly bounded and
\[
\begin{align*}
|x_n(t) - x_1(t)| &\leq \|K^*\| \|x(t) - x(t_1)\| + g(\mu) \|x\| \|k_{t_2} - k_{t_1}\|_q \|\varphi\|_p + \int_{t_1}^{t_2} k_{t_2}(s)\varphi(s) \, ds.
\end{align*}
\]

Thus
\[
|x_n(t_2) - x_n(t_1)| \left[ 1 - K^* \right] \leq \|K_{t_2} - K_{t_1}\|_q \|\varphi\|_p + \int_{t_1}^{t_2} k_{t_2}(s)\varphi(s) \, ds.
\]

That is
\[
|x_n(t_2) - x_n(t_1)| \frac{1}{1 - K^*} \left[b_{1,2} + [g(\mu)\gamma] k_{1,2}\right].
\]

The above estimate yields that the family of functions $\{x_n\}$ is equicontinuous on $[0, 1]$; hence, relatively compact on $[0, 1]$. Therefore, we can extract a uniformly convergent subsequence $\{x_{n_k}\}$, that is, there exists a decreasing sequence $\{n_k\}$ such that $x_{n_k} \rightarrow x$ as $n_k \rightarrow \infty$ and $\lim_{n_k \rightarrow \infty} x_{n_k}(t)$ exists uniformly in $t \in [0, 1]$; we denote this limiting value by $\psi(t)$.

Obviously, the uniform continuity of $g$ and $H(\cdot, x(\cdot))$ together with the equation
\[
x_n(t) = \frac{1}{n_k} + H(t, x_{n_k}(t)) + x_{n_k}(t) \int_0^t k(t, s)\varphi(s)x_{n_k}(s) \, ds, \quad t \in [0, 1],
\]
yields $\psi$ as a solution of Eq. (1). Next, we show that the solution $\psi$ is either a maximal or a minimal solution to Eq. (1). To do this, we observe problem (14) and we let $m, n \in \mathbb{N}$ with $m < n$. Keeping in mind that $u(0) = 0$, we obtain
\[ x_n(0) - x_m(0) = \frac{1}{n} - \frac{1}{m} + H(0, x_n(0)) - H(0, x_m(0)). \]  
(15)

So, we have $x_n(0) \neq x_m(0)$ for otherwise, we get $m = n$ which contradicts our hypothesis that $m < n$. Thus, we have one of the following possibilities
\[ x_n(0) > x_m(0) \quad \text{or} \quad x_n(0) < x_m(0). \]  
(16)

We start with the case $x_n(0) > x_m(0)$ and we will show, in this case, that $\psi$ is the minimal solution to Eq. (1). A similar argument with the case $x_n(0) < x_m(0)$, shows that $\psi$ is the maximal solution to Eq. (1).

First, we will show that if $x_n(0) > x_m(0)$, then
\[ x_n(t) > x_m(t) \quad \text{for all} \quad t \in [0, 1]. \]  
(17)

To prove conclusion (17), we assume that it is false; then there exist $a t_1 \in [0, 1]$ such that
\[ x_m(t_1) = x_n(t_1) \quad \text{and} \quad x_m(t) > x_n(t), \quad \text{for all} \quad t \in [0, t_1). \]

Since $g$ is monotonic nonincreasing in $x$, it follows, using Eq. (14), that
\[ x_m(t_1) = \frac{1}{m} + H(t_1, x_m(t_1)) + x_m(t_1) \int_0^{t_1} k(t_1, s) \psi(s) g(x_m(s)) \, ds \]
\[ > \frac{1}{n} + H(t_1, x_n(t_1)) + x_n(t_1) \int_0^{t_1} k(t_1, s) \psi(s) g(x_n(s)) \, ds \]
\[ = x_n(t_1), \]
which contradict the fact that $x_m(t_1) = x_n(t_1)$. Hence, inequality (17) is true.

Second, we show that $\psi$ is the minimal solution to Eq. (1).

To achieve this goal, we let $x$ be any solution of (1) existing on the interval $[0, 1]$. Then, $x_m(t) < x(t), \quad t \in [0, 1]$. Since the minimal solution is unique (see [11,12]), it is clear that $x_m(t)$ tends to $\psi(t)$ uniformly in $t \in [0, 1]$ as $m \to \infty$, which proves the existence of minimal solutions to the integral equation (1).

A similar argument making up the possibility that $x_n(0) < x_m(0)$, implies the existence of maximal solutions. This ends the proof. □

**Example 3.1.** In the view of Theorem 3.1 and Example 2.2, one can deduce that the following functional integral equation
\[ x(t) = \frac{1}{\ln(5)} \ln \left( \frac{1 + \sqrt{x(t)}}{1 + t} \right) + \frac{x(t)}{5} \int_0^t \frac{s}{1 + t + s^2} \left[ \frac{1}{4 \sqrt{x(s)}} \right] \, ds, \quad t \in [0, 1], \]  
(18)

has either maximal or minimal solutions in the space $C([0, 1], (0, \infty))$.

**Corollary 3.1.** Suppose that the assumptions of Theorem 3.1 are satisfied. If $H$ does not depend on $x$, then there exists a maximal solution to the integral equation (1) in the space $C([0, 1], (0, \infty))$.

**Proof.** Observe the proof of Theorem 3.1. To prove the existence of maximal solutions, it suffices to show that the inequality $x_n(0) < x_m(0)$, $m < n$ holds. To see this, we look at equality (15). Since $H$ is independent of $x$, we have
\[ H(0, x_n(0)) - H(0, x_m(0)) = 0. \]
This implies that the right-hand side of Eq. (15) (and consequently the left-hand side) is negative. That is, $x_n(0) - x_m(0) < 0, m < n$ holds. Thus, $x_n(0) < x_m(0), m < n$. Now we are able to repeat the rest of the proof of Theorem 3.1 to prove the existence of a maximal solution to the integral equation (1). □

Based on Corollary 3.1 and Example 2.4, the problem
\[ x(t) = e^{5+t} + x(t) \int_0^t \frac{t \exp \left( \frac{-1}{t+s} \right)}{(t+s)^2} \varphi(s) \left[ \ln \left( 1 + \frac{\sqrt{0.5x(s)}}{2x(s)} \right) \right] \, ds, \quad t \in [0, 1], \]
has maximal solution $x \in C([0, 1], (0, \infty))$.

In the following examples, we show that the singularity of $g$ at $x = 0$ gives the permission to discuss the solvability of some the Volterra-type integral equations as follows.
Example 3.2. Here, we consider the integral equation

\[ y^\vartheta(t) = 1 - \int_0^t k(t, s)\varphi(s)\Theta(y(s)) \, ds, \quad t \in [0, 1], \quad \beta > 0, \]

where \( \Theta : [0, \infty) \to [0, \infty) \) is a continuous and nondecreasing function. To see this, we assume that the assumptions of Theorem 2.1 hold with \( f \equiv 0, H \equiv 1 \) and \( g(x) = \Theta(x^{-\delta}), \delta > 0 \). In the view of Theorem 2.1, the integral equation

\[ x(t) = 1 + x(t) \int_0^t k(t, s)\varphi(s) \left( \frac{1}{x^2(s)} \right) \, ds, \quad t \in [0, 1], \]

has a positive solution \( x \in C[0, 1] \) with \( \mu \leq x(t) \leq \gamma, t \in [0, 1] \). Putting \( y = x^{-\delta}, \beta = 1/\delta \) implies that problem (19) has at least one positive solution \( y \in C[0, 1] \). Moreover, if \( k(t, s) \leq \vartheta(s), s \in [0, 1], \vartheta \in L^1 \) then, by Corollary 3.1, the problem (19) has a maximal solution.

Example 3.3. The following convolution-type integral equation

\[ \Psi(y(t)) = h_0 - \int_0^t P(t - s)y(s) \, ds, \quad t \in [0, 1], \]

with \( h_0 > 0, \ P \in L^1 \) and \( \Psi \in C[\mathbb{R}^+, \mathbb{R}^+] \) occurs in the theory of infiltration of a fluid from a reservoir into an isotropic porous medium (see [13]). To discuss problem (20), we consider functions \( \Theta, \ P \) with the following properties:

1. \( 0 \leq P_t(s) = P(t - s) \in L^1[0, 1] \) for each \( t \in [0, 1] \) and the map \( t \to P(t - s) \) is continuous from \([0, 1]\) to \( L^1[0, 1] \)
2. \( \Theta \in C[\mathbb{R}^+, \mathbb{R}^+] \) is locally expansive, that is, for any compact set \( M \subset (0, \infty) \), there is a constant \( m(M) > 0 \) such that

\[ |\Theta(x) - \Theta(y)| \geq m(M)|x - y|, \quad \text{for any } x, y \in M. \]

Observe that the conditions imposed on \( \Theta \) result in \( \Theta \) increasing and \( \Psi := \Theta^{-1} \in C[\mathbb{R}^+, \mathbb{R}^+] \).

Assume that the assumptions of Theorem 2.1 hold with \( f \equiv 0, H \equiv 1, \varphi \equiv \frac{1}{h_0} \) and \( g(x) = \Theta \left( \frac{1}{x} \right) \).

In the view of Theorem 2.1, the integral equation

\[ x(t) = \frac{1}{h_0} + \frac{x(t)}{h_0} \int_0^t P(t - s)\Theta \left( \frac{1}{x(s)} \right) \, ds, \quad t \in [0, 1], \]

has a positive solution \( x \in C[0, 1] \). Putting \( y(x) = \Theta \left( \frac{1}{x} \right) \) implies \( \frac{1}{x} = \Psi(y) \). Thus problem (20) has a positive solution \( y \in C[0, 1] \). Moreover, if \( P(t - s) \leq \vartheta(s), t, s \in [0, 1], \vartheta \in L^1 \) then, by Corollary 3.1, problem (20) has a maximal solution.

Remark 3.2. The conditions imposed on \( \Theta \) are quite general as they are satisfied by large classes of functions. A few examples of such functions are

\[ x \to x\ln(1 + x), \quad x \to x^p, \quad \text{with } p > 1, \]

as well as

\[ x \to x(e^x - 1), \quad x \to (e^x - 1)^2, \quad x \to x^2 e^x. \]

4. Application

Inspired by the application of the quadratic integral equation of fractional order in physics and motivated by the work of e.g. Banas [9], Darwish [10] and others, we will investigate, as a special case of problem (1), the following quadratic integral equation of fractional order

\[ x(t) = H(t, x(t)) + x(t)L^\alpha \varphi(s)(f(x(t)) + g(x(t))), \quad t \in [0, 1], \alpha \in (0, 1). \]

Here, \( L^\alpha \) denotes the fractional integral operator of order \( \alpha > 0 \).

Recall that the fractional integral operator of order \( \alpha > 0 \) with left-hand point 0 is defined by

\[ L^\alpha x(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} x(s) \, ds. \]

It is a well-known consequence of an inequality of Young that the linear fractional integral operators \( L^\alpha \) continuously into \( L^p([0, 1]) \) if \( p \in [1, \infty] \) satisfy \( q > 1/(\alpha + 1/p) \) (see [14]).
Theorem 4.1. Let α ∈ (0, 1), p > 1/α. Assume that the functions H, f, g and ϕ satisfy Assumptions (1)–(4) of Theorem 2.1. Apart from this, we assume the following hypothesis:

\[
2 \left( c_2(\gamma) + \frac{f(\gamma) + g(\mu)}{\Gamma(1 + \alpha)} \|\psi\|_p \right) \leq 1. \tag{22}
\]

Then Eq. (21) has at least one solution \( x \in C[0, 1] \) such that \( \mu \leq x(t) \leq \gamma, \ t \in [0, 1] \).

Proof. Define the kernel \( k \) by

\[
k(t, s) := \frac{1}{\Gamma(\alpha)} \begin{cases} (t - s)^{\alpha - 1}, & s < t, \\ 0, & \text{otherwise.} \end{cases}
\]

We will show that \( k \) satisfies the requirements of Theorem 4.1. To see this, let \( t, \tau \in [0, 1] \). Since \( p > 1/\alpha \), then we have by Hölder inequality with \( 1/p + 1/q = 1 \) and in the view of \( q(\alpha - 1) > -1 \) that

\[
\left( \int_0^1 |k(t, s)|^q \, ds \right)^{1/q} = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} ds \leq \frac{1}{\Gamma(\alpha) q(\alpha - 1) + 1}.
\]

Further, without loss of generality, we may assume that \( t < \tau \). Then, we obtain the following chain of inequalities

\[
\|k(t, s) - k(t, \tau)\|_q = \left( \int_0^1 |k(t, s) - k(t, \tau)|^q \, ds \right)^{1/q}
\]

\[
= \left( \int_0^t |k(t, s) - k(t, \tau)|^q \, ds + \int_0^\tau |k(t, s) - k(t, \tau)|^q \, ds + \int_t^\tau |k(t, s) - k(t, \tau)|^q \, ds \right)^{1/q}
\]

\[
= \frac{1}{\Gamma(\alpha)} \left[ \int_0^t |(t - s)^{\alpha - 1} - (\tau - s)^{\alpha - 1}|^q \, ds + \int_\tau^\tau (\tau - s)^{q(\alpha - 1)} \, ds + \int_\tau^t (\tau - s)^{q(\alpha - 1)} \, ds \right]^{1/q}
\]

\[
\leq \frac{1}{\Gamma(\alpha)} \left[ \int_0^t (\tau - s)^{q(\alpha - 1)} \, ds + \frac{(\tau - t)^{1 + q(\alpha - 1)}}{1 + q(\alpha - 1)} \right]^{1/q}
\]

\[
\leq C(\alpha, q) \left[ \tau - t \right]^{1 + q(\alpha - 1)} \left[ (\tau - t)^{1 + q(\alpha - 1)} ight]^{1/q}
\]

with some finite constant \( C(\alpha, q) \), depending only on \( \alpha \) and \( q \). Thus,

\[
\|k(t, s) - k(t, \tau)\|_q \leq C(\alpha, q) |\tau - t|^{1/(1/p)}.
\]

In summary, we have \( 0 \leq k(t, \cdot) \in L^p([0, 1]) \) for each \( t \in [0, 1] \) and the map \( k(\cdot, \cdot) \) is continuous from \( [0, 1] \) to \( L^q([0, 1]) \). Now observe, in the view of \( \|L^p \psi\|_p \leq \frac{1}{\Gamma(1 + \alpha)} \|\psi\|_p \) that inequality (22) is only a modified version of Assumption (6). Therefore, the result follows immediately from Theorem 2.1. This ends the proof. \( \square \)

Now we give an example illustrating Theorem 4.1.

Example 4.1. Consider the quadratic integral equation of the fractional type

\[
x(t) = \frac{1 + t}{10} + e^{-t} \frac{x^2(t)}{30} + \frac{x(t)}{10^{1/2}} \int_0^t (t - s)^{-0.5} \left[ \sqrt{x(s)} + \frac{1}{3 \sqrt{x(s)}} \right] \frac{1}{s^{1/6}} \, ds, \quad t \in [0, 1]. \tag{23}
\]

Observe that the above equation is a special case of Eq. (21) if we put \( \alpha = 1/2, \ p = 3 \) and

- \( H(t, x) = \frac{1 + t}{10} + e^{-t} \frac{x^2(t)}{30} \),
- \( \varphi(t) = \frac{1}{10^{1/2}} \),
- \( f(x) = \sqrt{x}, \ g(x) = \frac{1}{3 \sqrt{x}} \).
In what follows, we show that the functions involved in Eq. (23) satisfy inequality (22) of Theorem 4.1.

Indeed, by the results of Example 2.1 we have \( \mu = 0.1, \gamma = 0.3, c_2(\gamma) = 0.02; \) also we observe that \( \|\varphi\|_3 = 0.125 \cdots. \)

Therefore,

\[
2 \left( c_2(\gamma) + \frac{\sqrt{\gamma} + 1/(3\sqrt{\gamma})}{\Gamma(1.5)} \|\varphi\|_3 \right) = 2(0.02 + (1.82)0.216) \leq 0.83.
\]

Thus, the hypotheses of Theorem 4.1 are satisfied. Hence, we conclude that Eq. (23) has at least one solution \( x \in C[0, 1] \) such that \( 0.1 \leq x(t) \leq 0.3, \ t \in [0, 1]. \)

**Theorem 4.2.** If the assumptions of Theorem 4.1 are satisfied with \( f(\cdot) = 0 \) and with a constant function \( c_2, \) then there exists either maximal or minimal solutions of the integral equation (21) in the space \( C([0, 1], (0, \infty)). \)

**Proof.** The idea behind the proof is quite similar to the idea in the proof of Theorem 3.1 except that now we define \( u : [0, 1] \to \mathbb{R}^+ \) by

\[
u(t) := l^\alpha \varphi(t)g(x(t)).
\]

Since \( g(x(\cdot)) \) is continuous on \([0, 1] \) and \( \varphi \in L^p, \ p > (1/\alpha), \) we deduce, by the properties of the fractional calculus (see e.g. [15]), that \( u \in C([0, 1], \mathbb{R}^+). \) Moreover, we observe by Hölder inequality with \( 1/p + 1/q = 1 \) and in the view of \( q(\alpha - 1) > -1 \) that

\[
u(t) \leq \frac{g(\mu)\|\varphi\|_p}{\Gamma(\alpha)} \left[ \frac{t^{q(\alpha - 1) + 1}}{q(\alpha - 1) + 1} \right]^{(1/q)} = \frac{g(\mu)\|\varphi\|_p}{\Gamma(\alpha)} \left[ \frac{1}{q(\alpha - 1) + 1} \right]^{(1/q)} t^{\alpha - (1/p)}.
\]

Thus, \( u(t) \to 0 \) as \( t \to 0. \) Since \( u \) is continuous, one can deduce that \( u(0) = 0. \) Now we are able to follow the same arguments as in the proof of Theorem 3.1 to show that problem (21) has either maximal or minimal solutions. \( \square \)

As a consequence of Theorem 4.2, we introduce the following corollary. Since the arguments are only a slight variation of the arguments in the proof of Corollary 3.1, we skip the proof.

**Corollary 4.1.** Suppose that the assumptions of Theorem 4.2 are satisfied. If \( H \) does not depend on \( x, \) then there exists a maximal solution to the integral equation (21) in the space \( C([0, 1], (0, \infty)). \)

5. Quadratic integral equations of Hammerstein type

As a consequence of Theorem 2.1, we introduce the following theorems. The analysis is similar to that in the proof of this theorem; therefore, we skip the proof.

**Theorem 5.1.** If Assumptions (1)–(5) of Theorem 2.1 hold along with

\[
2 \left( c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{t \in [0, 1]} \int_0^1 k(t, s) \varphi(s) \, ds \right) \leq 1,
\]

then Eq. (3) has at least one solution \( x \in C[0, 1] \) such that \( \mu \leq x(t) \leq \gamma, \ t \in [0, 1]. \)

**Example 5.1.** Here, we need to show that the above results can be used so as to include the important quadratic integral equations of Chandrasekhar type

\[
x(t) = 1 + x(t) \int_0^1 \frac{t}{t + s} \varphi(s) x(s) \, ds.
\]

We remark, that usually the existence of solutions of the Chandrasekhar-type integral equation is derived under the additional assumption that the so-called characteristic function \( \varphi \) is an even polynomial in \( s, \) (cf., [5, Chapter 5]). For such characteristic functions, it is known that the resulting solutions can be expressed in terms of Chandrasekhar’s H-functions [5, Chapters 4 and 5].

In our case, we derive the existence of solutions of this equation under the much weaker assumption of continuity of \( \varphi. \) Indeed, we assume that \( \varphi \in L_{\infty}, \) that is \( \varphi \) is an essentially bounded function and need not to be continuous. Obviously, this equation is a particular case of Eq. (3), where \( H(\cdot, \cdot) = 1, \ f(x) = x, \ g(\cdot) = 0 \) and \( k(t, s) = \frac{1}{t + s}. \)

It is easy to check that, in this situation, the assumptions of Theorem 5.1 are satisfied with \( \mu = 1, \gamma \geq 2, \ c_1 = c_2 = 0. \) Indeed, it may be verified that:

\[
\sup_{t \in [0, 1]} \int_0^1 \frac{|\varphi(s)|}{t + s} \, ds \leq \|\varphi\|_{\infty} \ln 2.
\]
Thus, to ensure that inequality (24) will be valid, it is enough to take \( \varphi \in L_\infty \) such that
\[
\|\varphi\|_\infty < \frac{1}{(2 \ln 2)^\gamma}.
\]

Now Theorem 5.1 shows that the Chandrasekhar-type integral equation has at least one positive continuous solution in 
\( B_2 := \{ x \in C([0, 1]) : \|x\| \leq 2 \} \). Such a result, in the case of a general characteristic function, does not appear in any literature and so is new.

6. Quadratic integral equations on the half-line

Many authors have investigated the existence of solutions of integral equations on an unbounded interval with the help of some two-component measures of noncompactness in the Banach space \( BC[0, \infty) \). This approach seems to be too restrictive. In this paper, without any assumptions in terms of the measure of noncompactness, we extend the idea of Section 2 to obtain analogous results for singular quadratic integral equations of the form (5). By revising the conditions imposed on \( H, k, \varphi, f \) and \( g \) in Theorem 2.1, one can apply Schauder–Tychonoff Theorem to ensure that problem (5) has a continuous solution in the Fréchet space \( C[0, \infty) \). Conditions on \( H, k, \varphi, f \) and \( g \) further imply that the solutions belong to \( BC[0, \infty) \).

**Theorem 6.1.** Suppose that Assumptions (2) and (3) hold along with
\[
(1^0) \ H : [0, \infty) \times [0, \infty) \to [0, \infty) \text{ is a bounded continuous function and satisfies the following conditions:}
\]
\[
(a) \ 0 < \mu < \gamma \text{ exists such that}
\]
\[
H(t, x) \geq \mu \text{ holds for every } x \geq \mu \text{ and } \gamma \geq 2 \max_{t \in [0, \infty)} H(t, \mu),
\]
\[
(b) \text{ There exists a function } b \in BC[0, \infty) \text{ and a nondecreasing functions } c_i : [\mu, \infty) \to \mathbb{R}^+,
\]
\[
i = 1, 2, \text{ such that}
\]
\[
|H(t, x) - H(s, y)| \leq c_1(r)|b(t) - b(s)| + c_2(r)|x - y|,
\]
\[
\text{for all } x, y \in [\mu, r] \text{ and } t, s \in [0, \infty).
\]
\[
(2^0) \ 0 \leq \varphi \in L^p[0, \infty),
\]
\[
(3^0) \ 0 \leq k_i(s) = k(t, s) \in L^q[0, \infty) \text{ for each } t \in [0, \infty),
\]
\[
(4^0) \text{ the map } t \to k(t, s)\varphi(s) \text{ is continuous from } [0, \infty) \text{ to } L^1[0, \infty),
\]
\[
(5^0) \text{ the map } t \to k(t, s)\varphi(s) \text{ is bounded from } [0, \infty) \text{ to } L^1[0, \infty),
\]
\[
(6^0)
\]
\[
2\left(c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{t \in [0, \infty)} \int_0^\infty k(t, s)\varphi ds\right) \leq 1. \tag{26}
\]

Then problem (5) has at least one solution \( x \in C[0, \infty) \) such that \( \mu \leq x(t) \leq \gamma, t \in [0, \infty). \)

**Notation.** Throughout this section, we let
\[
b_{1,2}^* = c_1(y)|b(t_1) - b(t_2)|, \quad t_1, t_2 \in [0, \infty),
\]
\[
K^{**} = \left(c_2(\gamma) + (f(\gamma) + g(\mu)) \sup_{t \in [0, \infty)} \int_0^\infty k(t, s)\varphi(s) ds\right),
\]
and
\[
k_{1,2}^* = \int_0^\infty |k(t_2, s) - k(t_2, s)|\varphi(s) ds, \quad t_1, t_2 \in [0, \infty).
\]

It is easy to check that conditions (2^0)–(4^0) ensure that \((K_3)\) and \((K_4)\) are well defined.

**Proof.** The idea behind the proof is quite similar to the idea in the proof of Theorem 2.1 except that now we use the Schauder–Tychonoff fixed point theorem.

Define \( T \) as in the proof of Theorem 2.1 by
\[
Tx(t) := H(t, x(t)) + x(t) \int_0^\infty k(t, s)\varphi(s)(f(x(s)) + g(x(s))) ds, \quad t \in [0, \infty). \tag{27}
\]

Let the closed, convex subset \( C \), required by the Schauder–Tychonoff fixed point theorem, be given by
\[
C = \left\{ x \in BC[0, \infty) : \mu \leq x(t) \leq \gamma, \forall t \in [0, \infty) \text{ and } \forall t_1, t_2 \in [0, \infty) \right\}.
\]

We have
\[
|x(t_1) - x(t_2)| \leq \frac{1}{1 - K^{**}} \left[b_{1,2}^* + (f(\gamma) + g(\mu))\gamma k_{1,2}^*\right].
\]
First, note that \( T : C \to C \) is well-defined. Let \( x \in C \). Then \( Tx \in C(0, \infty) \) since for any \( t_1, t_2 \in [0, \infty) \), we have from Assumptions (1\(^3\))–(3\(^3\)) that

\[
|Tx(t_2) - Tx(t_1)| \leq c_1(||x||)|b(t_2) - b(t_1)| + |x(t_2) - x(t_1)| + c_2(||x||) \sup_{t \in (0, \infty)} \int_0^\infty k(t, s)\varphi(s) \, ds + ||x|| (f(||x||) + g(\mu)) \int_0^\infty |k(t_2, s) - k(t, s)|\varphi(s) \, ds
\]

and

\[
geq b_{1,2}^* + |x(t_2) - x(t_1)|k_{**} + \gamma (f(||x||) + g(\mu))k_{1,2}^* \to 0 \quad \text{as} \quad t_1 \to t_2,
\]

holds. In addition, it is clear from the definition of \( C \) that

\[
|Tx(t_1) - Tx(t_2)| \leq \frac{1}{1 - K_{**}} \left[ b_{1,2}^* + (f(\gamma) + g(\mu))\gamma k_{1,2}^* \right],
\]

for all \( x \in C(0, \infty) \), is true. Therefore, as in the proof of Theorem 2.1, it can be shown that \( T : C \to C \) is well-defined. Second, we show that \( T(C) \) is relatively compact in \( C \subseteq C(0, \infty) \). To do this we must show that \( T(C) \) is uniformly bounded and equicontinuous on each compact subinterval of \([0, \infty)\). Since \( T(C) \subseteq C \), it follows that \( T(C) \) is in fact uniformly bounded in \([0, \infty)\). Further, the equicontinuity of \( T(C) \) in each compact subinterval of \([0, \infty)\) follows directly from (28). Hence, \( T(C) \) (and also \( T(C(0, \infty)) \)) is relatively compact in \( C \subseteq C(0, \infty) \).

Finally, we require that \( T : C \to C \) is continuous. Suppose that \( x_0 \to x \) in \( C \subseteq C(0, \infty) \), that is, \( x_n \to x \) in \( C \subseteq C(0, m] \) for each \( m \in \{1, 2, \ldots \} \). Clearly this implies the pointwise convergence of \( x_n \) to \( x \) on \([0, \infty)\). Coupling this fact with Assumptions (2\(^3\))–(5\(^3\)), we obtain for each \( t \in [0, \infty) \) that

\[
H(t, x_n(t)) \to H(t, x(t)),
\]

\[
k_t(s)\varphi(s) \left[ f(x_n(t)) + g(x_n(t)) \right] \to k_t(s)\varphi(s) \left[ f(x(t)) + g(x(t)) \right], \quad \text{a.e.} \; s \in [0, \infty).
\]

Moreover for each \( t \in [0, \infty) \)

\[
0 \leq k_t(s)\varphi(s) \left[ f(x_n(t)) + g(x_n(t)) \right] \leq k_t(s)\varphi(s) \left[ f(\gamma) + g(\mu) \right], \quad \text{a.e.} \; s \in [0, \infty).
\]

Consequently, by the Lebesgue dominated convergence theorem we have

\[
Tx_n(t) \to Tx(t), \quad \text{for each} \; t \in [0, \infty) \; \text{as} \; n \to \infty.
\]

Now fix \( m \in \{1, 2, \ldots \} \). Since \([0, m] \) is compact, combining (28) and (29) yields

\[
Tx_n(t) \to Tx(t), \quad \text{in} \; C(0, m] \; \text{as} \; n \to \infty.
\]

Obviously, this is true for any \( m \in \{1, 2, \ldots \} \), and therefore

\[
Tx_n(t) \to Tx(t), \quad \text{in} \; C(0, m] \; \text{as} \; y_n \to y \; \text{in} \; C.
\]

In summary, we have that \( T : C \to C \) is a continuous and compact operator, and thus, by Schauder–Tychonoff fixed point theorem, \( T \) has a fixed point \( x \in C \) and we are finished.

\[\square\]

References