Stability analysis for Zakharov–Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma

A.R. Seadawy *
Mathematics Department, Faculty of Science and Arts, Taibah University, Al-Ula, Saudi Arabia
Mathematics Department, Faculty of Science, Beni-Suef University, Egypt

Abstract
The Zakharov–Kuznetsov (ZK) equation is an isotropic nonlinear evolution equation, first derived for weakly nonlinear ion-acoustic waves in a strongly magnetized lossless plasma in two dimensions. In the present study, by applying the extended direct algebraic method, we found the electric field potential, electric field and magnetic field in the form of traveling wave solutions for the two-dimensional ZK equation. The solutions for the ZK equation are obtained precisely and the efficiency of the method can be demonstrated. The stability of these solutions and the movement role of the waves are analyzed by making graphs of the exact solutions.

1. Introduction
The Zakharov–Kuznetsov (ZK) equation is a very attractive model equation for the study of vortices in geophysical flows. The ZK equation appears in many areas of physics, applied mathematics and engineering. In particular, it shows up in the area of Plasma Physics [1–3]. The ZK equation governs the behavior of weakly nonlinear ion-acoustic waves in a plasma comprised of cold ions and hot isothermal electrons in the presence of a uniform magnetic field [3–6].

The ZK equation [7] is one of two well-studied canonical two-dimensional extensions of the Korteweg–de Vries equation [8]. In contrast to the Kadomtsev–Petviashvili (KP) equation, the ZK equation has so far never been derived in a geophysical fluid dynamics context. For a derivation of the KP equation for internal waves, see [9].

Traveling wave analysis is given in [10] for the ZK equation. Soliton solutions are derived using the improved modified extended tanh-function method [10]. One-dimensional soliton, apparently inelastic [11], periodic solutions [12] and N-soliton solutions [13] have been obtained. The auxiliary equation method and the direct Hirota bilinear method were applied to the quantum ZK equation in [14–18].

This paper is organized as follows. In Section 2, the problem formulations to derive the nonlinear two-dimensional ZK equation are presented. In Section 3, the conservation laws for the ZK equation of weakly nonlinear ion-acoustic waves in a plasma are found. In Section 4, the Hamiltonian system for the momentum and the sufficient condition for soliton solution stability are given. In Section 5, the electric field potential, electric field and magnetic field in the form of traveling wave solutions of the ZK equation are obtained and analyzed. Finally the paper ends with a conclusion in Section 6.

2. Problem formulations

Consider a low-β plasma in a magnetic field $\mathbf{B} = B_e \mathbf{e}_n$, with $T_e \gg T_i$, where $T$ is the temperature and the subscripts $i$ and $e$ denote ions and electrons respectively. The ion motions are governed by

$$\frac{\partial n}{\partial t} + \nabla \cdot (nv) = 0,$$

where $n$ is the ion density and $v$ is the ion velocity.

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*Correspondence to: Mathematics Department, Faculty of Science, Beni-Suef University, Egypt.
E-mail address: Aly742001@yahoo.com.
\[
\frac{\partial v}{\partial t} + (v \cdot \nabla)v = - \frac{e}{m_i} \nabla \phi + v \wedge \Omega_i,
\]
\[
\nabla^2 \phi = -4\pi e(n - n_e),
\]
\[
n_e = n_0 \exp \left( \frac{e\phi}{kT} \right),
\]
where \( n \) is the number density of ions, \( v \) is their velocity, \( m_i \) is the mass of an ion, \( \phi \) is the electric field potential, and \( \Omega_i = \frac{eB}{m_i} \).

Let us non-dimensionalize the various quantities as follows:
\[
n' = \frac{n}{n_0}, \quad n'_e = \frac{n_e}{n_0}, \quad v' = \frac{v\sqrt{m_i}}{\sqrt{kT}}, \quad \phi' = \frac{\phi e}{kT},
\]
\[
x' = \frac{x}{\rho}, \quad z' = \frac{z}{L}, \quad t' = \frac{t\sqrt{m_i}}{L\sqrt{kT}}, \quad \rho = \frac{\sqrt{kT}e}{\Omega_i\sqrt{m_i}},
\]
where \( L \) is the scale length of the waves. The independent variables are
\[
\zeta = \sqrt{\epsilon}(z - t), \quad \eta = \sqrt{\epsilon}x, \quad \tau = \epsilon^{\frac{3}{2}}t,
\]
where \( \epsilon \) is the small parameter characterizing the typical amplitude of the waves, and using the reduction perturbation method, from Eqs. (1)-(4) the ZK equation can be derived as
\[
\frac{\partial \phi}{\partial \tau} + \frac{1}{2} \left( \frac{\partial \phi^2}{\partial \zeta} + \frac{\partial \phi^2}{\partial \eta} \right) + \frac{1}{2} (1 + \delta) \frac{\partial^3 \phi}{\partial \eta^2 \partial \zeta} = 0,
\]

where \( \delta = \frac{\lambda^2}{\rho^2}, \quad \lambda^2 = \frac{kT e}{4\pi n_0 e^2} \).

Consider the traveling wave solutions as
\[
\phi(\xi, \eta, \tau) = \phi(\xi), \quad \text{and} \quad \xi = k\zeta + \ell\eta + \omega\tau,
\]
where \( k, \ell \) and \( \omega \) are wave numbers and frequency. Then Eq. (7) becomes
\[
\omega \phi' + k\phi \phi' + \frac{1}{2} k^2 \phi^{(3)} + \frac{1}{2} k^2 (1 + \delta) \phi^{(3)} = 0.
\]

### 3. Conservation laws

The ZK equation possesses some polynomial conservation laws, which can be cast into the following conservation forms:
\[
\phi_{\tau} + \frac{1}{2} \left( \phi^2 + \phi_{\zeta} + (1 + \delta) \phi_{\eta\eta} \right)_{\zeta} = 0,
\]
\[
\left( \frac{1}{2} \phi^2 \right)_{\tau} + \frac{1}{12} \left( 4\phi^3 + 6\phi\phi_{\zeta} - 3\phi^2 \right)_{\tau} + \left( 4 + 6 + 3(1 + \delta) \phi\phi_{\eta\eta} + 3(1 + \delta) \phi^2 \right)_{\zeta} - \frac{1}{2} (1 + \delta) \phi_{\eta\eta} = 0.
\]

Thus, if the electric field potential vanishes sufficiently rapidly at the ends of some intervals, it is easy to show that three integrals of motion (conserved quantities) exist for Eq. (7):
\[
I_1[\phi] = \int \int \phi \, d\zeta \, d\eta,
\]
\[
I_2[\phi] = \frac{1}{2} \int \int |\psi|^2 \, d\zeta \, d\eta,
\]
\[
I_3[\phi] = \frac{1}{2} \int \int \left( 2\phi^3 + \left| \frac{\partial \phi}{\partial \zeta} \right|^2 + \left| \frac{\partial \phi}{\partial \eta} \right|^2 \right) \, d\zeta \, d\eta.
\]

These integrals of motion were also found by Infeld [11] using a variational formulation, and can provide rigorous information about the outcome of collisions of ZK solitary waves.

### 4. Stability analysis

Eq. (7) is a Hamiltonian system for which the momentum is given by
\[
M = \frac{1}{2} \int_{-\infty}^{\infty} \phi^2 \, d\xi
\]
where $M$ is the momentum and $\phi$ is the electric field potential. The sufficient condition for soliton stability is
\[
\frac{\partial M}{\partial \omega} > 0
\]
where $\omega$ is the frequency.

5. Exact traveling wave solutions

Now we shall find classes of solutions of the two-dimensional ZK equation, by applying the direct algebraic function method. The different values for $\phi'$ give different analytic solutions of Eq. (7), which gives the following five cases.

Case I. Assume that the ZK equation (9) has the following formal solution:
\[
\phi(\xi) = \sum_{i=0}^{m} a_i \psi^i(\xi), \quad \text{and} \quad (\psi')^2 = \alpha \psi^2 + \beta \psi^4.
\] (17)

where $\alpha, \beta$ are arbitrary constants. Balancing the nonlinear term $\phi \psi'$ and the highest order derivative $\phi^{(3)}$ in Eq. (9) gives $m = 2$. The solution of Eq. (9) is in the form
\[
\phi(\xi) = a_0 + a_1 \psi + a_2 \psi^2.
\] (18)

Substituting from (18) into Eq. (9) yields a set of algebraic equations for $a_0, a_1, a_2, k, \ell, \omega, \alpha, \beta, \delta$. The system of equations are found as
\[
k^3 a_1 + k \ell^2 a_1 + k \ell^2 \alpha + 2 \omega a_1 + 2ka_0 a_1 = 0,
ka_1^2 + k^4 a_2 + 4 k \ell^2 \alpha a_2 + 4 k \ell^2 \alpha \delta a_2 + 2 \omega a_2 + 2ka_0 a_2 = 0,
k^4 \beta a_1 + k \ell^2 \beta a_1 + k \ell^2 \beta \delta a_1 - ka_1 a_2 = 0,
12 k^2 \beta a_2 + 12 k \ell^2 \beta a_2 + 12 k \ell^2 \beta \delta a_2 - 2ka_2 = 0.
\] (19)

The solution for the system of Eqs. (19), can be found as
\[
a_0 = -\frac{2k^3 \alpha + 2k \ell^2 \alpha + 2k \ell^2 \alpha \delta + \omega}{k}, \quad a_1 = 0, \quad a_2 = 6\beta(k^2 + \ell^2 + \ell^2 \delta).
\] (20)

Substituting from Eq. (20) into (18), the electric field potential of Eq. (7) can be obtained as a bell-shaped solitary wave solution
\[
\phi(\xi, \eta, \tau) = -\frac{2k^3 \alpha + 2k \ell^2 \alpha + 2k \ell^2 \alpha \delta + \omega}{k} + 6\beta \alpha(k^2 + \ell^2(1 + \delta)) \operatorname{sech}^2(\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau)).
\] (21)

The electric field $\vec{E} = -\nabla \phi = -\frac{\alpha \psi}{\alpha \xi} \hat{\xi} - \frac{\alpha \psi}{\alpha \eta} \hat{\eta}$;
\[
\vec{E} = -12\alpha^{3/2}(k^2 + \ell^2(1 + \delta)) \operatorname{sech}^2(\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau)) \tanh(\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau))(k\vec{\xi} + \ell\vec{\eta}).
\] (22)

From Eq. (3), the difference between the number density of ions and electrons can be obtained as
\[
n - n_e = -\frac{1}{4\pi e} \nabla^2 \phi
= -\frac{3}{\pi e} \alpha^2(k^2 + \ell^2)(k^2 + \ell^2(1 + \delta)) \operatorname{sech}^4(\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau))(-2 + \cosh(2\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau))).
\] (23)

From the Maxwell–Faraday equation
\[
\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial \tau}
\]
the magnetic field can be obtained in the form
\[
\vec{B} = \frac{24}{\omega} \alpha^{3/2} k \ell(k^2 + \ell^2(1 + \delta)) \operatorname{sech}^2(\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau)) \tanh(\sqrt{\alpha}(k\xi + \ell\eta + \omega\tau)) \hat{\eta}.
\] (24)

The electric field potential solution (21) is a Hamiltonian system for which the momentum is given by
\[
M = \frac{1}{2} \int_{-5}^{5} \phi^2 d\xi.
\] (25)

The sufficient condition for soliton solution stability is
\[
\frac{\partial M}{\partial \omega} > 0.
\] (26)
The magnetic field (24) is represented by the solitary waves shown in Fig. 1(c) and the electric field is shown in Fig. 1(d) with \( k = 1, \ell = 0.5, \alpha = 1, \beta = 1.5, \delta = 0.3, \omega = 2, \gamma = 0.5 \), in the interval \([-5, 5]\). According to the conditions of stability (25)–(26), the electric field potential (21), the electric field (22) and the magnetic field (24) are stable in the interval \([-5, 5]\).

Case II. Suppose that the solution of the ZK equation (7) has the following form:

\[
\phi(\xi) = \sum_{i=0}^{m} a_i \varphi^i(\xi), \quad \text{and} \quad (\varphi')^2 = \alpha \varphi^2 + \beta \varphi^3 + \gamma \varphi^4, \tag{27}
\]

where \( \gamma \) is arbitrary constant. Balancing the nonlinear term \( \phi \varphi' \) and the highest order derivative \( \phi^{(3)} \) in Eq. (9) gives \( m = 2 \). The solution of Eq. (9) is of the form

\[
\phi(\xi) = a_0 + a_1 \varphi + a_2 \varphi^2. \tag{28}
\]

Substituting from (28) into Eq. (9) yields a set of algebraic equations for \( a_0, a_1, a_2 \). The solution of the system of these algebraic equations can be found as

\[
a_0 = -\frac{\alpha}{2} (k^2 + \ell^2 + \ell^2 \delta) - \frac{\omega}{k}, \quad a_1 = -3\beta (k^2 + \ell^2 + \ell^2 \delta), \quad a_2 = -\frac{3\beta^2}{2\alpha} (k^2 + \ell^2 + \ell^2 \delta), \quad \gamma = \frac{\beta^2}{4\alpha}. \tag{29}
\]

Substituting from Eq. (29) into (28), the electric field potentials can be obtained as the following exponential solutions of Eq. (7):

\[
\phi_1(\zeta, \eta, \tau) = -\frac{\alpha}{2} (k^2 + \ell^2 + \ell^2 \delta) - \frac{\omega}{k} - \frac{12\beta (k^2 + \ell^2 + \ell^2 \delta) \exp(k\zeta + \ell\eta + \omega\tau)}{(\exp(k\zeta + \ell\eta + \omega\tau) - \beta)^2 - 4\gamma} \frac{24\gamma (k^2 + \ell^2 + \ell^2 \delta) \exp(2(k\zeta + \ell\eta + \omega\tau))}{\alpha (\exp(k\zeta + \ell\eta + \omega\tau) - \beta)^2 - 4\gamma^2}, \tag{30}
\]

\[
\phi_2(\zeta, \eta, \tau) = -\frac{\omega}{k} - \frac{12\beta (k^2 + \ell^2 + \ell^2 \delta) \exp(k\zeta + \ell\eta + \omega\tau)}{1 - 2\beta \exp(k\zeta + \ell\eta + \omega\tau) + (\beta^2 - 4\gamma) \exp(2(k\zeta + \ell\eta + \omega\tau))} \frac{-\alpha}{2} (k^2 + \ell^2 + \ell^2 \delta) - \frac{24\gamma (k^2 + \ell^2 + \ell^2 \delta) \exp(2(k\zeta + \ell\eta + \omega\tau))}{\alpha (1 - 2\beta \exp(k\zeta + \ell\eta + \omega\tau) + (\beta^2 - 4\gamma) \exp(2(k\zeta + \ell\eta + \omega\tau)))}. \tag{31}
\]

The electric fields \( \vec{E} \) can be obtained as

\[
\vec{E}_1 = \frac{12f(\zeta, \eta, \tau) (f^2 - \beta^2 + 4\gamma) (f^2 \beta + \beta^3 - 2f(\beta^2 - 8\gamma) - 4\beta \gamma) (k^2 + \ell^2 (1 + \delta))}{(f(\zeta, \eta, \tau) - \beta)^2 - 4\gamma^2} (\vec{E}_\zeta + \ell \vec{E}_\eta), \tag{32}
\]
and

\[
\vec{E}_2 = \frac{12f ((\beta^2 - 4\gamma) f^2 - 1) (\beta + (\beta^2 - 4\gamma) \beta f^2 - 2(\beta^2 - 8\gamma) f) (k^2 + \ell^2 (1 + \delta))}{(1 - 2\beta f + (\beta^2 - 4\gamma) f^2)^3} (\vec{E}_\zeta + \ell \vec{E}_\eta), \tag{33}
\]
where \( f = f(\xi, \eta, \tau) = \exp(k\xi + \ell\eta + \omega\tau) \). From the Maxwell–Faraday equation, the magnetic field can be obtained in the form

\[
\vec{B}_1 = \left( \frac{3k\ell(k^2 + \ell^2(1 + \delta)) \left( 2\beta^2(\beta^2 - 4\gamma)^3 - 2f(\beta^2 - 4\gamma)^3(5\beta^2 - 4\gamma) - 2bf^5(\beta^2 + 20\gamma) \right)}{2\omega(f - \beta^2 - 4\gamma)^3} \right)
\times (2f^4(5\beta^4 + 52\gamma\beta^2 - 192\gamma^2) - 4f^3\beta(5\beta^4 + 12\gamma\beta^2 - 96\gamma^2) + 4f^2(\beta^2 - 4\gamma)(5\beta^4 + 32\gamma^2))
\]

\[
+ \frac{3k\ell\beta}{2\omega\gamma\sqrt{\beta}} (\beta^2 - 4\gamma)(k^2 + \ell^2(1 + \delta)) \tanh^{-1} \left( \frac{f - \beta}{2\sqrt{\gamma}} \right) \vec{e}_n
\]

\[
\vec{B}_2 = \left( \frac{12k\ell(k^2 + \ell^2(1 + \delta)) \left( 2f(\beta - \beta f^4(\beta^2 - 4\gamma)^2 - 2f(\beta^2 - 8\gamma) + 2f^3(\beta^2 - 4\gamma)(\beta^2 - 8\gamma)) \right)}{\omega(1 - 2f\beta + (\beta^2 - 4\gamma)f^2)^3} \right)
\times \frac{(k^2 + \ell^2(1 + \delta)) \tanh^{-1} \left( \frac{\beta - (\beta^2 - 4\gamma)f}{2\sqrt{\gamma}} \right)}{\omega\sqrt{\gamma}} \vec{e}_n.
\]

The bright solitary wave solution and contour plot for the electric field potential (30)–(31) are shown in Figs. 2(a)–(d), with \( k = 0.5, \ell = -1, \alpha = 4.5, \beta = -3, \delta = 2, \omega = -0.1, \tau = 0.5, \gamma = 0.5; k = 2, \ell = -1, \alpha = 0.5, \beta = -3, \delta = 2, \omega = -1, \tau = 3, \gamma = 5 \) in the interval \([-5, 5]\) and \([-10, 10]\). The electric field (32) is shown in Fig. 2(e) and the magnetic field (34) is represented solitary wave solution in Fig. 2(f) with \( k = 1, \ell = 2, \alpha = 0.5, \beta = -1.4, \delta = -1, \omega = 0.3, \tau = 0.5, \gamma = 5 \) in the interval \([-2, 2]\) and \([-10, 10]\). According to the conditions of stability (25)–(26), the electric field potential (30)–(31), the electric field (32)–(33) and the magnetic field (34)–(35) are stable in the interval \([-10, 10]\); \([-5, 5]\); \([-2, 2]\).

Case III. Assume that the solution of the ZK equation (9) is in the following form:

\[
\phi(\xi) = a_0 + a_1\phi + a_2\phi^2 + a_3\phi^3 + a_4\phi^4, \quad (\phi')^2 = a\phi^4 + \beta\phi^5
\]

where \( a_0, a_1, a_2, a_3, a_4 \) are arbitrary constants. Substituting from (36) into Eq. (9) yields a set of algebraic equations for \( a_0, a_1, a_2, a_3, a_4, k, \ell, \omega, \alpha, \beta, \delta, \gamma \). The solution of the system of these algebraic equations can be found as

\[
a_0 = -\frac{\omega}{k}, \quad a_1 = 0, \quad a_2 = -6\alpha(k^2 + \ell^2 + \ell^2\delta), \quad a_3 = 0, \quad a_4 = -24\beta(k^2 + \ell^2 + \ell^2\delta).
\]

Substituting from Eq. (37) into (36), the following solution of Eq. (7) can be obtained for the electric field potential in the form of rational solution as

\[
\phi(\xi, \eta, \tau) = -\frac{\omega}{k} \frac{6\alpha^2(k^2 + \ell^2 + \ell^2\delta)}{\alpha^2(k\xi + \ell\eta + \omega\tau)^2 - \beta} \pm \frac{24\alpha^2\beta(k^2 + \ell^2 + \ell^2\delta)}{(\alpha^2(k\xi + \ell\eta + \omega\tau)^2 - \beta)^2}.
\]

The electric fields \( \vec{E} \) can be obtained as

\[
\vec{E} = \frac{12\alpha^3 \left( (k^2 + \ell^2(1 + \delta))(k\xi + \ell\eta + \omega\tau)(-\beta\alpha + 8\beta + \alpha^2(k^2 + \ell^2\eta + \omega\tau)^2)\right)}{(\alpha^2(k\xi + \ell\eta + \omega\tau)^2 - \beta)^3} (k\vec{e}_\xi + \ell\vec{e}_\eta).
\]
From the Maxwell–Faraday equation, the magnetic field can be obtained in the form

\[
\mathbf{B} = \frac{6\alpha^2 k\ell(k^2 + \ell^2(1 + \delta)) \left( \xi (-5\alpha\gamma^2 + \gamma\alpha^5\xi^4 + 4\alpha\gamma(8\beta + \alpha^2\xi^2)) \right)}{\gamma \omega (\alpha^2\xi - \gamma)^3} \\
- \frac{6\alpha^2 k\ell(k^2 + \ell^2(1 + \delta))}{\omega\gamma} \tanh^{-1} \left( \frac{\alpha\xi}{\sqrt{\gamma}} \right) \mathbf{\hat{e}_\theta}. \tag{40}
\]

The bright solitary wave solution and contour plot for the electric field potential (38) are shown in Fig. 3(a)–(b), with \( k = 0.2, \ell = 1.5, \alpha = 2, \beta = 1.5, \delta = -1.02, \omega = -1, \tau = 2, \gamma = 1 \) in the interval \([-5, 5] \) and \([-10, 10] \).

Fig. 2(a-b). Electric field potential (30): (a) bright solitary waves, (b) contour plot, in the interval \([-10, 10]\).

Fig. 2(c-d). Electric field potential (31): (c) bright solitary waves, (d) contour plot, in the intervals \([-5, 5]\) and \([-10, 10]\).

Fig. 2(e-f). The electric field and magnetic field: vector plot in 2(e), solitary wave solution in 2(f), in the intervals \([-2, 2]\) and \([-10, 10]\).
Substituting from Eq. (42) into (41), the solution of Eq. (7) can be obtained as

\[ a_0 = -\frac{2\alpha k^3 + 2\alpha k^2(1 + \delta) + \omega}{k}, \quad a_1 = 0, \quad a_2 = -6\beta(k^2 + \ell^2(1 + \delta)). \]  

Substituting from Eq. (42) into (41), the solution of Eq. (7) can be obtained as

\[ \phi_1(\xi, \eta, \tau) = -\frac{2\alpha k^3 + 2\alpha k^2(1 + \delta) + \omega}{k} \pm \frac{12\beta(\alpha k^2 + \alpha \ell^2(1 + \delta))e^{2(k\xi + \ell\eta + n\tau)}}{\beta^2 - 4\gamma - 2\beta e^{2(k\xi + \ell\eta + n\tau)}}, \]

\[ \phi_2(\xi, \eta, \tau) = -\frac{2\alpha k^3 + 2\alpha k^2(1 + \delta) + \omega}{k} \pm \frac{12\beta(k^2 + \ell^2(1 + \delta))e^{2(k\xi + \ell\eta + n\tau)}}{1 - 2\beta e^{2(k\xi + \ell\eta + n\tau)} + (\beta^2 - 4\gamma)e^{4(k\xi + \ell\eta + n\tau)}}. \]

The electric fields \( \vec{E} \) can be obtained as

\[ \vec{E}_1 = \frac{48\beta e^{2(k\xi + \ell\eta + n\tau)}(4\beta - \gamma^2 + e^{4(k\xi + \ell\eta + n\tau)})(k^2 + \ell^2(1 + \delta))}{\left( -4\beta + (\gamma - e^{2(k\xi + \ell\eta + n\tau)})^2 \right)}(k\vec{e}_\xi + \ell\vec{e}_\eta). \]

\[ \vec{E}_2 = \frac{48\beta e^{2(k\xi + \ell\eta + n\tau)}(\gamma^2 - 4\beta e^{4(k\xi + \ell\eta + n\tau)} - 1)(k^2 + \ell^2(1 + \delta))}{(1 - 2\gamma e^{2(k\xi + \ell\eta + n\tau)} + (\gamma^2 - 4\beta)e^{4(k\xi + \ell\eta + n\tau)})^2}(k\vec{e}_\xi + \ell\vec{e}_\eta). \]

From the Maxwell–Faraday equation, the magnetic field can be obtained in the form

\[ \vec{B}_1 = \frac{48\beta e^{2(k\xi + \ell\eta + n\tau)}(\gamma^2 - 4\beta - e^{4(k\xi + \ell\eta + n\tau)} - 1)(k^2 - \ell^2)(k^2 + \ell^2(1 + \delta))}{\omega(\gamma^2 - 4\beta - 2\gamma e^{2(k\xi + \ell\eta + n\tau)} + e^{4(k\xi + \ell\eta + n\tau)})^2}\vec{e}_\eta. \]

Figs. 4(a)–(d), represent the traveling wave solutions (43)–(44) of the ZK equation (7) and contour plot, with \( k = 0.5, \ell = -1.5, \beta = 1, \delta = 0.8, \omega = -1, \tau = 0.5, \gamma = 1, \alpha = 1 \) and \( k = 1.5, n = -0.5, \beta = 1, \delta = 1, \omega = 1, \tau = 0.5, \gamma = 3 \) in the intervals \([-5, 5]\) and \([-4, 4]\).

Case V. The solution of the ZK equation (9) has the following form:

\[ \phi(\xi) = a_0 + a_1\phi + a_2\phi^2, \quad \phi' = \alpha + \beta\phi + \gamma\phi^2. \]

Substituting from (48) into Eq. (9) yields a set of algebraic equations for \( a_0, a_1, a_2, k, \ell, \omega, \alpha, \beta, \gamma, \delta \). The solution of the system of these algebraic equations can be found as

\[ a_0 = -\frac{1}{2}\alpha(\alpha + 8\beta)(k^2 + \ell^2(1 + \delta)) - \frac{\omega}{k}, \quad a_1 = -6\alpha\beta(k^2 + \ell^2(1 + \delta)), \quad a_2 = -6\beta^2(k^2 + \ell^2(1 + \delta)). \]
Substituting from Eq. (49) into (48), the solution of Eq. (7) can be obtained in the form of the Riccati sub-equation method as
\[
\phi(\zeta, \eta, \tau) = -\frac{1}{2} \alpha (\alpha + 8 \beta) (k^2 + \ell^2 (1 + \delta)) - \frac{\omega}{k} \\
+ 6 \alpha \beta (k^2 + \ell^2 (1 + \delta)) \left( \frac{\beta}{2\gamma} - \frac{\sqrt{4\alpha \gamma - \beta^2}}{2\gamma} \tan \left( \sqrt{\alpha \gamma - \beta^2/4} (k \zeta + \ell \eta + \omega \tau) \right) \right) \\
- 6 \beta^2 (k^2 + \ell^2 (1 + \delta)) \left( -\frac{\beta}{2\gamma} + \frac{\sqrt{4\alpha \gamma - \beta^2}}{2\gamma} \tan \left( \sqrt{\alpha \gamma - \beta^2/4} (k \zeta + \ell \eta + \omega \tau) \right) \right)^2.
\]
(50)

The electric fields $\vec{E}$ can be obtained as
\[
\vec{E} = -\frac{3\beta}{2\gamma^2} (4\alpha \gamma - \beta^2) (k^2 + \ell^2 (1 + \delta)) \left( \sec \left( \frac{1}{2} \sqrt{4\alpha \gamma - \beta^2} (k \zeta + \ell \eta + \omega \tau) \right) \right)^2 \\
\times \left( \gamma \alpha - \beta^2 + \beta \sqrt{4\alpha \gamma - \beta^2} \tan \left( \frac{1}{2} \sqrt{4\alpha \gamma - \beta^2} (k \zeta + \ell \eta + \omega \tau) \right) \right) (k \vec{\varepsilon}_x + \ell \vec{\varepsilon}_y).
\]
(51)

From the Maxwell–Faraday equation, the magnetic field can be obtained in the form
\[
\vec{B} = \frac{k \ell \beta}{\gamma^2 \omega} (4\alpha \gamma - \beta^2) (k^2 + \ell^2 (1 + \delta)) \left( \tan \left( \frac{1}{2} \sqrt{4\alpha \gamma - \beta^2} (k \zeta + \ell \eta + \omega \tau) \right) \right)^2 \\
\times \left( 3 \gamma \alpha - 3 \beta^2 + \beta \sqrt{4\alpha \gamma - \beta^2} \tan \left( \frac{1}{2} \sqrt{4\alpha \gamma - \beta^2} (k \zeta + \ell \eta + \omega \tau) \right) \right) \vec{\varepsilon}_\eta.
\]
(52)

Both dark solitary wave solutions and contour plot (50) are shown in Fig. 5(a)-(b), with $k = 0.5, \ell = 1.5, \beta = 1, \delta = 3, \omega = -1, \tau = 2, \gamma = 1$ in the intervals $[-1, 1]$ and $[-3, 3]$. 

Fig. 4(a-b). The electric field potential (43): solitary wave solution in 3(a), contour plot in 3(b), in the intervals $[-5, 5]$ and $[-4, 4]$.

Fig. 4(c-d). The electric field potential (44): solitary wave solution in 4(c), contour plot in 4(d), in the intervals $[-5, 5]$ and $[-4, 4]$.
6. Conclusion

The problem formulations of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field is given for the nonlinear two-dimensional ZK equation. By implementing the extended direct algebraic method, we present traveling wave solutions for the nonlinear two-dimensional ZK equation. The stability analysis for the electric field potentials, electric fields, and magnetic fields are discussed with respect to the sufficient condition for soliton stability. We obtained in case I the electric field potential in the form of a bell-shaped solitary wave solution. The electric field potential, electric field, and magnetic field are stable in the interval $[-5, 5]$. In cases II and IV, we obtained solutions in the form of exponential solitary wave solutions and these solutions are different in form and coefficients from the Ji-Huan He solutions. The solutions in form rational solitary wave solutions are given in case III. In case V, we obtained the electric field potential in the form of dark solitary wave solution which is stable in the interval $[-1, 1]$.

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References